ON OPTIMAL DISCRETE CORRECTION OF THE FORCED MOTION OF STOCHASTIC SYSTEMS

(OB OPTIMAL'NOI DISKRETNOI KORREKTSII VYNUZHDENNOGO DVIZHENIIA STOKHASTICHESKIKH SISTEM)

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The problem of synthesizing the optimal control of the finite state of a linear stochastic system is considered. The problem reduces to the solution of Bellman's functional equations. Bellman's equations are solved with reference to the problem of guidance toward a specified phase point.

1. Statement of the problem. Let us consider the controlled system described by the equations

$$\frac{dy}{dt} + A(t)\,y = B(t)\,f(t), \qquad f(t) = w(t) + q(t) \tag{1.1}$$

Here y is the n-vector of the phase coordinates y_i; $A = \{a_{kj}\}$ and $B = \{b_{ir}\}$ are certain matrices of dimensions $n \times n$ and $n \times s$, respectively, w is the s-vector function of Markov-type random perturbations and q is the q-vector of the discrete controlling signals q_k.

The control is understood to be discrete in the sense that the magnitudes of the controlling signals over the given correction intervals $[t_i,\ t_{i+1})$ are determined at the initial instant $t_i\ (i = 1, \ldots, \nu)$ of the interval.

The association of the variable quantity with the instant t_i will be denoted by the superscript i.

It is assumed that the system (1.1) is completely controllable [1 and 2], that the phase coordinates y_i are measurable, and that the a priori distribution of perturbations w_k is known.

Our problem is to determine on the basis of information about the instantaneous values of the phase coordinate vector $y(t)$ a control $q = q(y)$ which, by the specified instant $t_{\nu+1}$, brings the system (1.1) from the state $y(t_1) = y^1$ to some state $y(t_{\nu+1}) = y^{\nu+1}$ under the condition of minimization of the mathematical expectation of the prescribed positive function $\omega(y^{\nu+1})$. Thus, our task is that of finding the control which minimizes the functional

$$I = \langle \omega\,(y^{\nu+1}) \rangle \tag{1.2}$$

The angle brackets here denote the mathematical expectation.

Formulated this way, our problem is related to that of the analytical construction of a regulator [3]. The stochastic aspect of the problem has been dealt with in several papers, of which [4] and [5] may be noted here.

2. Discretization of the process. Within the time interval under consideration, the general solution of system (1.1) is of the form

$$y\,(t) = N\,(t,\, t_1)\, y^1 + \int_{t_1}^{t} N\,(t,\, \tau)\, B\,(\tau)\, f\,(\tau)\, d\tau \tag{2.1}$$

Here $N\,(t,\, \tau) = Y\,(t)\, Y^{-1}\,(\tau)$ is the matrix function of the weight of system (1.1), Y is the fundamental matrix of the homogeneous equation of (1.1) normalized for $t = t_1$ and Y^{-1} is the inverse matrix.

Let us suppose that we know the points $y^i = (y_1^{\,i}, \ldots, y_n^{\,i})$ and $y^{i+1} = (y_i^{i+1}, \ldots, y_n^{i+1})$, through which the phase trajectory (2.1) in one of the actual realizations passes at the instants t_i and t_{i+1}, respectively, representing the end points of the i-th correction interval.

On the basis of (2.1), let us express the vector of the final state of system (1.1) as a function of y^i and y^{i+1}. We obtain

$$y^{\nu+1} = N\,(t_{\nu+1},\, t_i)\, y^i + \int_{t_i}^{t_{\nu+1}} N\,(t_{\nu+1},\, \tau)\, B\,(\tau)\, f\,(\tau)\, d\tau \tag{2.2}$$

$$y^{\nu+1} = N\,(t_{\nu+1},\, t_{i+1})\, y^{i+1} + \int_{t_{i+1}}^{t_{\nu+1}} N\,(t_{\nu+1},\, \tau)\, B\,(\tau)\, f\,(\tau)\, d\tau \tag{2.3}$$

Subtracting (2.3) from (2.2) we find, that

$$N\,(t_{\nu+1},\, t_{i+1})\, y^{i+1} = N\,(t_{\nu+1},\, t_i)\, y^i + \int_{t_i}^{t_{i+1}} N\,(t_{\nu+1},\, \tau) \Big[B\,(\tau)\, w\,(\tau) + \sum_{k=1}^{s} B_k\,(\tau)\, q_k\,(\tau) \Big] d\tau \qquad (i = 1, \ldots, \nu) \tag{2.4}$$

Here B_k is the k-th column vector of the matrix B. Further, we introduce the vectors

$$m^{i+1} = N\,(t_{\nu+1},\, t_{i+1})\, y^{i+1} + \int_{t_{i+1}}^{t_{\nu+1}} N\,(t_{\nu+1},\, \tau)\, \langle w\,(\tau) \rangle\, d\tau \qquad (i = 0, \ldots, \nu) \tag{2.5}$$

$$\varepsilon^j = \int_{t_j}^{t_{j+1}} N\,(t_{\nu+1},\, \tau)\, B\,(\tau)\, [w\,(\tau) - \langle w\,(\tau) \rangle]\, d\tau, \quad Q_k^{\,j} = \int_{t_j}^{t_{j+1}} N\,(t_{\nu+1},\, \tau)\, B_k\,(\tau)\, q_k\,(\tau)\, d\tau$$

$$(j = 1, \ldots, \nu;\ k = 1, \ldots, s) \tag{2.6}$$

Here $\langle w\rangle = (\langle w_1\rangle, \ldots, \langle w_s\rangle)$ is the mathematical expectation of the random vector function w.

By means of the vectors and relation (2.4) the finite state control process can be represented as a Markov chain by virtue of the discreteness of the controlling signals.

In fact, substituting (2.4) into (2.5) and taking into account Expression (2.6), we obtain

$$m^{i+1} = m^i + \varepsilon^i + \sum_{k=1}^{s} Q_k{}^i \qquad (i = 1, \ldots, \nu) \tag{2.7}$$

which defines (since $y^{\nu+1} = m^{\nu+1}$) ν, i.e. the stepwise process of variation of the finite state as determined by the random vectors ε^i and by the controlling signals $Q_k{}^i\,(q_k)$.

By virtue of the random character of the vectors ε^i, transformation (2.7) associates with specific values of m^i and Q_k^i some set of random realizations of the vector m^{i+1}. The law of distribution of this set, apart from the values of the vectors m^i and Q_k^i, depends on the distribution of the random vector ε^i.

3. Bellman's equations. In order to determine the optimal control for system (1.1) we turn to the method of dynamic programming [6]. Here we assume that the controlling signals q_k belong to the class of functions for which the relation

$$Q_k{}^i = \int_{t_i}^{t_{i+1}} N(t_{\nu+1}, \tau)\, B_k(\tau)\, q_k(\tau)\, d\tau = H_k{}^i u_k{}^i \qquad (i = 1, \ldots, \nu;\ \ k = 1, \ldots, s) \tag{3.1}$$

is fulfilled.

Here H_k^i is a certain vector which is independent of q_k and u_k^i are the control parameters constituting the vector $u^i = (u_1{}^i, \ldots, u_s{}^i)$.

With consideration of (3.1), relation (2.7) becomes

$$m^{i+1} = m^i + \varepsilon^i + \sum_{k=1}^{s} H_k{}^i u_k{}^i \qquad (i = 1, \ldots, \nu) \tag{3.2}$$

Since $y^{\nu+1} = m^{\nu+1}$, the problem of synthesizing the optimal control of system (1.1) consists in finding the sequence of vector functions $u^i = u^i\,(m^i)$ $(i = 1, \ldots, \nu)$, which optimizes Markov process (3.2) in the sense of minimization of criterion (1.2).

Following the method of dynamic programming, we introduce the notation

$$\Omega_k\,(m^j) = \min_{u^i} I = \min_{u^i} \langle \omega\,(y^{\nu+1})\rangle \qquad (k = 1, \ldots, \nu;\ \ j = \nu + 1 - k)$$

Here $\Omega_k\,(m^j)$ is the minimum value of the criterion in a process consisting of k steps and beginning with the state m^j. Minimization is effected with respect to the vector controls $u^i = u^i\,(m^i)$, and the mathematical expectation is computed from the set of random vectors ε^i $(i = j, \ldots, \nu)$.

Finding the optimal control then reduces to the solution of Bellman's functional $\Omega_k\,(m^j)$ and $\Omega_{k-1}\,(m^{j+1})$. For the process under consideration, Bellman's equations are of the form

$$\Omega_1(m^\nu) = \min_{u^\nu} \langle \omega(m^{\nu+1}(m^\nu))\rangle, \qquad \Omega_k(m^j) = \min_{u^j} \langle \Omega_{k-1}(m^{j+1}(m^j))\rangle \tag{3.3}$$

$$(k = 2, \ldots, \nu; \quad j = \nu + 1 - k)$$

Here the transformation $m^{j+1} = m^{j+1}(m^j)$ is defined in accordance with (3.2).

4. The problem of guidance towards the origin. Let us consider the solution of Equations (3.3) with reference to the problem of optimally precise guidance of system (1.1) towards the origin. As the measure of closeness of the end point of the phase trajectory of motion (1.1) to the origin of the coordinate system we take the function $\omega = (y^{\nu+1})^2$. Equations (3.3) can then be written as

$$\Omega_1(m^\nu) = \min_{u^\nu} \langle (m^{\nu+1}(m^\nu))^2\rangle, \quad \Omega_k(m^j) = \min_{u^j} \langle \Omega_{k-1}(m^{j+1}(m^j))\rangle \tag{4.1}$$

$$(k = 2, \ldots, \nu; \quad j = \nu + 1 - k)$$

Let us turn to the first equation of system (4.1). Expressing $m^{\nu+1}$ in terms of m^ν in accordance with transformation (3.2), we find that

$$\Omega_1(m^\nu) = \min_{u^\nu} \left\langle (m^\nu + \varepsilon^\nu + \sum_{\alpha=1}^{s} H_\alpha{}^\nu u_\alpha{}^\nu)^2 \right\rangle \tag{4.2}$$

Bearing in mind that ε^ν is a vector of random centered quantities, upon transformation of the right-hand side of (4.2) we obtain

$$\Omega_1(m^\nu) = \min_{u^\nu} \left(m^\nu + \sum_{\alpha=1}^{s} H_\alpha{}^\nu u_\alpha{}^\nu\right)^2 + \langle(\varepsilon^\nu)^2\rangle \tag{4.3}$$

The problem of finding the minimum with respect to $u^\nu = (u_1{}^\nu, \ldots, u_s{}^\nu)$ in (4.3) reduces to the solution of the system of linear equations

$$\begin{aligned}
&(H_1{}^\nu \cdot H_1{}^\nu)\, u_1{}^\nu + (H_1{}^\nu \cdot H_2{}^\nu)\, u_2{}^\nu + \cdots + (H_1{}^\nu \cdot H_s{}^\nu)\, u_s{}^\nu = -m^\nu \cdot H_1{}^\nu \\
&(H_2{}^\nu \cdot H_1{}^\nu)\, u_1{}^\nu + (H_2{}^\nu \cdot H_2{}^\nu)\, u_2{}^\nu + \cdots + (H_2{}^\nu \cdot H_s{}^\nu)\, u_s{}^\nu = -m^\nu \cdot H_2{}^\nu \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&(H_s{}^\nu \cdot H_1{}^\nu)\, u_1{}^\nu + (H_s{}^\nu \cdot H_2{}^\nu)\, u_2{}^\nu + \cdots + (H_s{}^\nu \cdot H_s{}^\nu)\, u_s{}^\nu = -m^\nu \cdot H_s{}^\nu
\end{aligned} \tag{4.4}$$

Here and below the dot is used to denote the scalar multiplication of vectors.

The solution of system (4.4) determines the optimal control $u^{o\nu}$ in the final step of the final step of the correction process,

$$u_\alpha{}^{o\nu} = -\sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^\nu}{\Delta^\nu} (m^\nu \cdot H_\beta{}^\nu) \qquad (\alpha = 1, \ldots, s) \tag{4.5}$$

Here $A_{\beta\alpha}^\nu$ is the algebraic complement of the element $(H_\beta{}^\nu \cdot H_\alpha{}^\nu)$ of the determinant $\Delta^\nu \neq 0$ of system (4.4).

Substituting (4.5) into (4.3), we find the minimum value of the optimality criterion in a one-step process,

$$\Omega_1(m^\nu) = \left(m^\nu - \sum_{\alpha=1}^{s} H_\alpha{}^\nu \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^\nu}{\Delta^\nu} (m^\nu \cdot H_\beta{}^\nu)\right)^2 + \langle(\varepsilon^\nu)^2\rangle \tag{4.6}$$

We shall show by induction that the sequence of optimal controls u^{oj} in the ν-step process as determined by the solution of Equations (4.1) is formed in accordance with the law

$$u_\alpha{}^j = -\sum_{\beta=1}^{s} \frac{A^j_{\beta\alpha}}{\Delta^j} (m^j \cdot R_\beta{}^j) \qquad (j = 1, \ldots, \nu; \quad \alpha = 1, \ldots, s) \tag{4.7}$$

while the minimum value of the criterion is

$$\Omega_\nu (m^1) = \Big(m^1 - \sum_{i=1}^{\nu} \sum_{\alpha=1}^{s} R_\alpha{}^i \sum_{\beta=1}^{s} \frac{A^i_{\beta\alpha}}{\Delta^i} (m^1 \cdot R_\beta{}^i)\Big)^2 + \Big\langle \sum_{r=1}^{\nu} (\psi^r)^2 \Big\rangle$$
$$\Delta^i = \det | r^i_{\beta\alpha} | \neq 0, \qquad r^i_{\beta\alpha} = R_\beta{}^i \cdot R_\alpha{}^i \qquad (i = 1, \ldots, \nu; \quad \alpha, \beta = 1, \ldots, s) \tag{4.8}$$

Here $A_{\beta\alpha}^{\ i}$ is the algebraic complement of the term $r_{\beta\alpha}^{\ i}$ in the determinant Δ^i. The vectors R^i_α are given by the recurrent relation

$$R_\alpha{}^\nu = H_\alpha{}^\nu, \; R_\alpha{}^{\nu-j} = H_\alpha{}^{\nu-j} - \sum_{\varkappa=0}^{j-1} \sum_{\gamma=1}^{s} R_\gamma{}^{\nu-\varkappa} \sum_{\beta=1}^{s} \frac{A^{\nu-\varkappa}_{\beta\gamma}}{\Delta^{\nu-\varkappa}} (H_\alpha{}^{\nu-j} \cdot R_\beta{}^{\nu-\varkappa}) \qquad \begin{pmatrix} j = 1, \ldots, \nu - 1 \\ \alpha = 1, \ldots, s \end{pmatrix} \tag{4.9}$$

The random vectors ψ^r are formed with the aid of a relation analogous in structure to (4.9),

$$\psi^\nu = \varepsilon^\nu, \qquad \psi^{\nu-j} = \varepsilon^{\nu-j} - \sum_{\varkappa=0}^{j-1} \sum_{\gamma=1}^{s} R_\gamma{}^{\nu-\varkappa} \sum_{\beta=1}^{s} \frac{A^{\nu-\varkappa}_{\beta\gamma}}{\Delta^{\nu-\varkappa}} (\varepsilon^{\nu-j} \cdot R_\beta{}^{\nu-\varkappa}) \qquad (j = 1, \ldots, \nu - 1) \tag{4.10}$$

Lemma 4.1. Recurrent relation (4.9), where H^i_α are arbitrary vectors of the n-dimensional Euclidean space generates the set of vectors $R_\alpha{}^i$ ($i = 1, \ldots, \nu$; $\alpha = 1, \ldots, s$), in which all the vectors with different superscripts are pairwise orthogonal.

Proof. We must show that

$$R^{\nu-x}_\alpha \cdot R^{\nu-y}_\varepsilon = 0 \qquad (x = 1, \ldots, \nu - 1; \; y = 0, \ldots, x - 1; \; \alpha, \varepsilon = 1, \ldots, s) \tag{4.11}$$

The validity of (4.11) for $x = 1$ directly verifiable. The proof of identities (4.11) for any x can be carried out by induction. We shall show that if Equations (4.11) are valid for $x = 1, \ldots, k$, then they are also valid for $x = k + 1$.

In accordance with (4.9), for the scalar product of the vectors $R_\alpha^{\ \nu-(k+1)}$ and $R_\varepsilon^{\ \nu-y}$ we have

$$R^{\nu-(k+1)}_\alpha \cdot R^{\nu-y}_\varepsilon = H^{\nu-(k+1)}_\alpha \cdot R^{\nu-y}_\varepsilon - \sum_{\varkappa=0}^{k} \sum_{\gamma=1}^{s} (R^{\nu-\varkappa}_\gamma \cdot R^{\nu-y}_\varepsilon) \sum_{\beta=1}^{s} \frac{A^{\nu-\varkappa}_{\beta\gamma}}{\Delta^{\nu-\varkappa}} (H^{\nu-(k+1)}_\alpha \cdot R^{\nu-\varkappa}_\beta) \tag{4.12}$$

Under our assumption as regards the validity of (4.11) for $x = 1, \ldots, k$, relation (4.12) becomes

$$R^{\nu-(k+1)}_\alpha \cdot R^{\nu-y}_\varepsilon = H^{\nu-(k+1)}_\alpha \cdot R^{\nu-y}_\varepsilon - \Big(\sum_{\beta=1}^{s} R^{\nu-y}_\beta \sum_{\gamma=1}^{s} \frac{r^{\nu-y}_{\varepsilon\gamma} A^{\nu-y}_{\beta\gamma}}{\Delta^{\nu-y}}\Big) \cdot H^{\nu-(k+1)}_\alpha \tag{4.13}$$

Since, in accordance with the properties of the algebraic complements of the determinant,

$$\sum_{\gamma=1}^{s} \frac{r_{\varepsilon\gamma}^{\nu-y} A_{\beta\gamma}^{\nu-y}}{\Delta^{\nu-y}} = \begin{cases} 1 & (\beta = \varepsilon) \\ 0 & (\beta \neq \varepsilon) \end{cases}$$

it follows on the basis of (4.13) that we have the required expression

$$R_{\alpha}^{\nu-(k+1)} \cdot R_{\varepsilon}^{\nu-y} = 0$$

The induction is now complete and the lemma has been proved.

Now let us prove the validity of expressions (4.7) and (4.8) for any ν.

We shall show that if a relation of the form (4.8) is valid for a k-step optimal process which begins with the state m^j, then it is also valid for an optimal process consisting of $k+1$ steps. Here the control $u^{\circ j-1}$, which carries the system from the state m^{j-1} the state m^j, must be formed in accordance with the law (4.7).

For a k-step optimal process let us have

$$\Omega_k(m^j) = \Big(m^j - \sum_{i=j}^{\nu} \sum_{\alpha=1}^{s} R_\alpha{}^i \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^i}{\Delta^i}(m^j \cdot R_\beta{}^i)\Big)^2 + \Big\langle \sum_{r=j}^{\nu} (\psi^r)^2 \Big\rangle \qquad (j = \nu + 1 - k) \tag{4.14}$$

Replacing m^j in (4.14) by its expression in accordance with (3.2) and taking account of (4.9) and (4.10), we obtain

$$\Omega_k(m^{j-1}) = \Big(m^{j-1} - \sum_{i=j}^{\nu} \sum_{\alpha=1}^{s} R_\alpha{}^i \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^i}{\Delta^i}(m^{j-1} \cdot R_\beta{}^i) + \sum_{\gamma=1}^{s} R_\gamma{}^{j-1} u_\gamma{}^{j-1} + \psi^{j-1}\Big)^2 + \Big\langle \sum_{r=j}^{\nu} (\psi^r)^2 \Big\rangle \tag{4.15}$$

In accordance with (4.1), for the optimal control we have

$$\Omega_{k+1}(m^{j-1}) = \min_{u^{j-1}} \langle \Omega_k(m^{j-1}) \rangle \tag{4.16}$$

Since the mathematical expectation of the vector ψ^{j-1} is equal to zero, substitution of (4.15) into (4.16) yields

$$\Omega_{k+1}(m^{j-1}) = \min_{u^{j-1}} \Big(m^{j-1} - \sum_{i=j}^{\nu} \sum_{\alpha=1}^{s} R_\alpha{}^i \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^i}{\Delta^i}(m^{j-1} \cdot R_\beta{}^i) + \sum_{\gamma=1}^{s} R_\gamma{}^{j-1} u_\gamma{}^{j-1}\Big)^2 + \Big\langle \sum_{r=j-1}^{\nu} (\psi^r)^2 \Big\rangle \tag{4.17}$$

The vector control $u^{\circ j-1}$, which minimizes the right-hand side of (4.17), is given by the solution of the system of equations

$$\Big(m^{j-1} + \sum_{\gamma=1}^{s} R_\gamma{}^{j-1} u_\gamma{}^{j-1}\Big) \cdot R_\varepsilon{}^{j-1} - \sum_{i=j}^{\nu} \sum_{\alpha=1}^{s} (R_\alpha{}^i \cdot R_\varepsilon{}^{j-1}) \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^i}{\Delta^i}(m^{j-1} \cdot R_\beta{}^i) = 0 \qquad (\varepsilon = 1, \ldots, s)$$

which, taking account of the orthogonality of the vector R_s^{j-1} relative to the vectors R_α^i (Lemma 4.1), becomes

$$\begin{aligned} r_{11}^{j-1}u_1^{j-1} + r_{12}^{j-1}u_2^{j-1} + \cdots + r_{1s}^{j-1}u_s^{j-1} &= -m^{j-1}\cdot R_1^{j-1} \\ r_{21}^{j-1}u_1^{j-1} + r_{22}^{j-1}u_2^{j-1} + \cdots + r_{2s}^{j-1}u_s^{j-1} &= -m^{j-1}\cdot R_2^{j-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ r_{s1}^{j-1}u_1^{j-1} + r_{s2}^{j-1}u_2^{j-1} + \cdots + r_{ss}^{j-1}u_s^{j-1} &= -m^{j-1}\cdot R_s^{j-1} \end{aligned} \qquad (4.18)$$

Setting $\Delta^{j-1} \neq 0$, and solving system (4.18), we find that

$$u_\gamma^{\circ j-1} = -\sum_{\beta=1}^{s} \frac{A_{\beta\gamma}^{j-1}}{\Delta^{j-1}} (m^{j-1}\cdot R_\beta^{j-1}) \qquad (\gamma = 1, \ldots, s) \qquad (4.19)$$

Comparing (4.19) with (4.7), we see that control (4.19) is an element of sequence (4.7).

By substituting (4.19) into (4.17) we find the expression for the minimum value of the criterion in a $(k + 1)$-step optimal process,

$$\Omega_{k+1}(m^{j-1}) = \left(m^{j-1} - \sum_{i=j-1}^{\nu}\sum_{\alpha=1}^{s} R_\alpha^i \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^i}{\Delta^i}(m^{j-1}\cdot R_\beta^i)\right)^2 + \left\langle \sum_{r=j-1}^{\nu} (\psi^r)^2 \right\rangle$$

which is analogous in form to (4.14).

Since the validity of formulas (4.14) and (4.19) for $k = 1$ has been demonstrated (see (4.5) and (4.6)), by induction they are also valid for any k, including $k = \nu$. Thus, the validity of expressions (4.7) and (4.8) has been proved.

5. A special case. For mechanical systems, the dimensionality n of the phase coordinate space is double the dimensionality s of the force vector space. In this important special case, the following theorem is valid.

Theorem 5.1. If $n = 2s$ and if the vectors $H_1^{\nu-1}, \ldots, H_s^{\nu-1}$, and $H_1^\nu, \ldots, H_s^\nu$, corresponding to two last correction intervals $[t_{\nu-1}, t_\nu)$ and $[t_\nu, t_{\nu+1})$, are linearly independent, then the minimum value of the criterion $I = \langle (y^{\nu+1})^2 \rangle$ is independent of the course of the process within the interval $[t_1, t_{\nu-1})$.

Proof. For $H_1^{\nu-1}, \ldots, H_s^{\nu-1}$, and $H_1^\nu, \ldots, H_s^\nu$ which are linearly independent, recurrent relation (4.9) generates n linearly independent vectors $R_1^{\nu-1}, \ldots, R_s^{\nu-1}$, and $R_1^\nu, \ldots, R_s^\nu$, forming the basis of the n-dimensional vector space R_β^j. Since, by Lemma 4.1, the vectors R_α^i ($\alpha = 1, \ldots, s$; $i = 1, \ldots, \nu - 2$) are orthogonal to the basis vectors, they can only be null vectors. Because of this, at the instants $t_1, \ldots, t_{\nu-2}$ system (4.18) defining the optimal control degenerates into the vector equation

$$0u^i = 0 \qquad (i = 1, \ldots, \nu - 2)$$

which is fulfilled for any u^i.

Hence, the minimum value of the criterion I is independent of the course of the process in the interval $[t_1, t_{\nu-1})$. The theorem has been proved.

Let us find the expression for the minimum value of the optimality criterion in the case at hand.

Since $R_\alpha{}^i = 0$ $(\alpha = 1, \ldots, s;\ i = 1, \ldots, \nu - 2)$, relation (4.8), taking account of (4.10), becomes

$$\Omega_\nu(m^1) = \min_u I = \Big[m^1 - \sum_{\alpha=1}^{s} R_\alpha{}^{\nu-1} \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^{\nu-1}}{\Delta^{\nu-1}} (m^1 \cdot R_\beta{}^{\nu-1}) - \tag{5.1}$$

$$- \sum_{\alpha=1}^{s} R_\alpha{}^\nu \sum_{\beta=1}^{s} \frac{A_{\beta\alpha}^{\nu}}{\Delta^\nu} (m^1 \cdot R_\beta{}^\nu)\Big]^2 + \Big\langle (\varepsilon^\nu)^2 + \Big(\varepsilon^{\nu-1} - \sum_{\gamma=1}^{s} R_\gamma{}^\nu \sum_{\beta=1}^{s} \frac{A_{\beta\gamma}^{\nu}}{\Delta^\nu} (\varepsilon^{\nu-1} \cdot R_\beta{}^\nu)\Big)^2 \Big\rangle$$

It is easy to show that the coefficients of the basis vectors $R_1{}^{\nu-1}, \ldots R_s{}^{\nu-1}$, $R_1{}^\nu, \ldots, R_s{}^\nu$ in the square bracket in the right-hand side of (5.1) are the coordinates of the n-vector m^1 relative to the indicated basis.

Hence, for the minimum value of the criterion we obtain

$$\min_u I = \Big\langle (\varepsilon^\nu)^2 + \Big(\varepsilon^{\nu-1} - \sum_{\gamma=1}^{s} R_\gamma{}^\nu \sum_{\beta=1}^{s} \frac{A_{\beta\gamma}^{\nu}}{\Delta^\nu} (\varepsilon^{\nu-1} \cdot R_\beta{}^\nu)\Big)^2 \Big\rangle$$

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